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LETTER TO THE EDITOR

Geometric foundations of a new conservation law discovered by Hojman

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Abstract. The geometric foundations of a recent new conservation law obtained by Hojman, for second-order differential equations, are given.

In a recent paper [1], Hojman obtained a new conservation law holding for a set of second-order differential equations

$$\ddot{q}_i = f_i(t, q, \dot{q}) \qquad i = 1, \ldots, n \tag{1}$$

where the forces f_i satisfy:

$$\sum_{i=1}^{n} \frac{\partial f_i}{\partial \dot{q}_i} + \boldsymbol{X}(\ln \lambda(q)) = 0 \tag{2}$$

and $\boldsymbol{X}$ is the vector-field associated canonically to (1), given by:

$$\boldsymbol{X} = 1 \cdot \frac{\partial}{\partial t} + \sum_{i=1}^{n} \left[\dot{q}_i \frac{\partial}{\partial q_i} + f_i \cdot \frac{\partial}{\partial \dot{q}_i} \right] \tag{3}$$

and $\lambda(q)$ is a function of $q_1, \ldots, q_n$.

Under the above hypothesis, and assuming that a symmetry vector

$$s = \sum_{i=1}^{n} \xi_i(f, q, \dot{q}) \frac{\partial}{\partial q_i}$$

of equations (1) is known, Hojman obtained a new conservation law for system (1), namely:

$$\sum_{i=1}^{n} \left[\frac{1}{\lambda} \frac{\partial(\lambda \xi_i)}{\partial q_i} + \frac{1}{\lambda} \frac{\partial}{\partial \dot{q}_i} (\lambda \boldsymbol{X}(\xi_i)) \right]. \tag{4}$$

That is, formula (4) is constant along trajectories of (1).

We present here the geometrical basis of this conservation law. In doing it we shall see that Hojman's hypothesis can be broadened, in order to include the case in which λ is a function of t, q, $\dot{q}$, and not only of the q as Hojman did.

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Consider the local volume form given by:

$$\Omega = dt \wedge dq_l \wedge \ldots \wedge dq_n \wedge d\dot{q}_1 \wedge \ldots \wedge d\dot{q}_n \tag{5}$$

and the Lie derivatives operator L_Y along the streamlines of an arbitrary vector-field $\boldsymbol{Y}$ [2].

Using Ω and L_Y we define the divergence of a vector-field Y in the usual form:

$$L_Y\Omega \stackrel{\text{def}}{=} (\operatorname{div} Y)\Omega = \sum_{i=1}^{2n+1} \frac{\partial Y_i}{\partial x_i} \qquad x = (t, q, \dot{q}). \tag{6}$$

In particular, when $\boldsymbol{Y}$ is the vector-field $\boldsymbol{X}$ defined in (3) we have:

$$L_X\Omega = \sum_{i=1}^{n} \frac{\partial f_i}{\partial \dot{q}_i} \tag{7}$$

and therefore we can write (2) in the form:

$$L_X\Omega + \boldsymbol{X}(\ln\lambda) \cdot \Omega = 0. \tag{8}$$

On the other hand, it is well known that if $\boldsymbol{S}$ is a symmetry vector of (1) we have:

$$[\boldsymbol{X}, \boldsymbol{S}^1] = 0 \tag{9}$$

where $\boldsymbol{S}^1$ is the first extension of $\boldsymbol{S}$, given by:

$$\boldsymbol{S}^1 = \boldsymbol{S} + \sum_{i=1}^{n} \frac{\partial}{\partial \dot{q}_i}[\boldsymbol{X}(\xi_i)] \tag{10}$$

and where [,] stands for the Lie–Jacobi bracket of vector-fields.

Now, from (9) one immediately has:

$$L_X L_{S^1} \cdot \Omega = L_{S^1} L_S \cdot \Omega \tag{11}$$

and therefore

$$L_X(\operatorname{div} \boldsymbol{S}^1) \cdot \Omega = L_{S^1}(\operatorname{div} \boldsymbol{X}) \cdot \Omega = L_{S^1}(-\boldsymbol{X}(\ln\lambda)) = L_X(-L_{S^1}(\ln\lambda)) \tag{12}$$

which can be written as:

$$L_X(\operatorname{div} \boldsymbol{S}^1 + L_{S^1}(\ln\lambda)) = 0 \tag{13}$$

that is,

$$\operatorname{div}(\boldsymbol{S}^1) + L_{S^1}(\ln\lambda) \tag{14}$$

is a new conservation law for $\boldsymbol{X}$.

Taking (10) into account it can be immediately checked that (14) is the conservation law obtained by Hojman.

Note that when obtaining the conservation law (14) we have not assumed that λ is a function of only $q_1, \ldots, q_n$, as Hojman did. Our reasoning is valid, in fact, when λ is a function of t, q, $\dot{q}$. Therefore the conservation law obtained here holds under a slightly broader hypothesis than those considered by Hojman.

References

[1] Hojman S A 1992 *J. Phys. A: Math. Gen.* **25** 291
[2] Abraham R and Marsden 1978 *Foundations of Mechanics* 2nd edn (Menlo Park: Benjamin Cummings)